Lecture 1

Definitions:

COORDINATE AXIS: $x, y, z-$ axis, perpendicular to eachother, through O
COORDINATE PLANES: 3 options:

(1) $xy-$plane:	(2) $xz-$plane:	(3) $yz-$plane:
contains $x-$ and $y-$ axis.	:contains $x-$ and $z-$ axis.	contains $y-$ and $z-$ axis.

OCTANTS: the eight parts in space, divided by the coordinate planes.
FIRST OCTANT: determined by the positive axes.
Point P has the ordered triple (a, b, c) where COORDINATES a, b, c: a =x-coordinate,b =y-coordinate&c = z-coordinate.
PROJECTION OF P: when projection on $xz-$ plane,$y-$ coordinate equals 0, works same way for $yz-$ and $xy-$ plane.
THREE-DIMENSIONAL RECTANGULAR COORDINATE SYSTEM: system where one-to-one correpsondence between a point and ordered triplets $(a, b, c) \in \mathbb{R}^3$
SURFACE IN $\mathbb{R}^3$: in 3d analytic geometry, an equation in x, y, z

DISPLACEMENT VECTOR V denoted by $\mathbf{v}$ or $\vec{v}$ the vector repreesents the movement along a line segment.
INITIAL POINT: tail of vector and TERMINAL POINT: the tip. Write $\mathbf{v} = \vec{AB}$
$\mathbf{u} = \mathbf{v}$ EQUIVALENT OR EQUAL: same length, same direction, same possition not necessory.
ZERO VECTOR $\mathbf{0}$ length 0
$\vec{AC} = \vec{AB} + \vec{AC}$

New formula's

Distance formula in three dimensions: distance $|P_1P_2|$ between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is:
$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
Equation of a sphere: Equation sphere with center $C(h, k, l)$ and radius r:
$(x - h)^2 + (y - k)^2_(z - l)^2 = r^2$
When center=O then: $x^2 + y^2 + z^2 = r^2$

Algebra vectors (1):

Definition of vector addition: $\mathbf{u}$&$\mathbf{v}$ vectors possitioned s.t. initial point $\mathbf{v}$ = terminal point $\mathbf{v}$ then $\mathbf{u}+\mathbf{v}$ vector initial point $\mathbf{u}$ to terminal point $\mathbf{v}$
Parallelogram Law: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
SCALER: a real number with which we multiply something. In this case a vector.
Definition scaler multiplication: c scaler $\mathbf{v}$ vector then: (1) scaler multiple $c\mathbf{v}$ vector whose length $|c|$ times length of $\mathbf{v}$
(a) Same direction as $\mathbf{v}$ if $c > 0$
(b) opposite if $c < 0$
(c) $c = 0$ or $\mathbf{v} = 0$ then $c\mathbf{v} = 0$
PARALLEL: two vectors if scaler multiples one another.
NEGATIVE of $\mathbf{v}$ same length as $\mathbf{v}$ opposite direction: $-\mathbf{v} = (-1)\mathbf{v}$
DIFFERENCE $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$

Components:

terminal **a** @origin, then coordinates called COMPONENTS: $\mathbb{R}^2$ $\langle a_1, a_2 \rangle$ $\mathbb{R}^3$ $\langle a_1, a_2, a_3 \rangle$

REPRESENTATIONS: gives an image of a vector.

vector representation: $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ then $\overrightarrow{AB} = \mathbf{a} = \langle x - 2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

POSITION VECTOR OF POINT P: $\overrightarrow{OP}$

Length of magnitude: $\mathbf{v}$ denoted by $|\mathbf{v}|$ or $\|\mathbf{v}\|$ the length of any representations:

$\mathbb{R}^2$ $\mathbf{a} = \langle a_1, a_2 \rangle$ $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$

$\mathbb{R}^3$ $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Algebra vectors (2):

$\mathbf{a} = \langle a_1, a_2 \rangle \& \mathbf{b} = \langle b_1, b_2 \rangle$ then:

(-) $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$

(-) $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$

(-) $c\mathbf{a} = \langle ca_1, ca_2 \rangle$

$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \& \mathbf{b} = \langle b_1, b_2, b - 3 \rangle$

(-) $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

(-) $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$

(-) $c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$

Properties of vectors: $\mathbf{a}, \mathbf{b}, \mathbf{c}$ vectors in V_n and α, β scalers:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \qquad \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$
$$\mathbf{a} + \mathbf{0} = \mathbf{a} \qquad \mathbf{a} + (-\mathbf{a}) = \mathbf{0}$$
$$\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b} \qquad (\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$$
$$(\alpha\beta)\mathbf{a} = \alpha(\beta\mathbf{a}) \qquad 1\mathbf{a} = \mathbf{a}$$

Definitions:

STANDARD BASIS VECTORS: $\mathbf{i}, \mathbf{j}, \mathbf{k}$ where $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$ and $\mathbf{k} = \langle 0, 0, 1 \rangle$

If $\mathbf{a} = \langle a_1, a_2 \rangle$ then $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ then $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

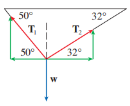
UNIT VECTOR: vector length 1. For example $\mathbf{i}, \mathbf{j} \& \mathbf{k}$

if $\mathbf{a} \neq \mathbf{0}$ then unit vector same direction as $\mathbf{a}$ is: $\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$

applications:

RESULTANT FORCE: the sum of the forces experienced by the object.

Example: 100-lb weight. Find $\mathbf{T}_1 \& \mathbf{T}_2$ and the magnitudes.



From this figure, we see that:

$\mathbf{T}_1 = -|\mathbf{T}_1| \cos(50°)\mathbf{i} + |\mathbf{T}_1| \sin(50°)\mathbf{j}$

$\mathbf{T}_2 = -|\mathbf{T}_2| \cos(32°)\mathbf{i} + |\mathbf{T}_2| \sin(32°)\mathbf{j}$

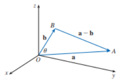
$\mathbf{T}_1 + \mathbf{T}_2 = \mathbf{w} = -100\mathbf{j}$

After some algebra we find that $|\mathbf{T}_1| \frac{100}{\sin 50° + \tan 32° \cos 50°}$ and $|\mathbf{T}_2| = \frac{|\mathbf{T}_1| \cos 50^{\text{deg}}}{\cos 32^{\text{deg}}}$

And $\mathbf{T}_1 \approx -55.06\mathbf{i} + 65.60\mathbf{j} \& \mathbf{T}_2 \approx 55.05\mathbf{i} + 34.40\mathbf{j}$

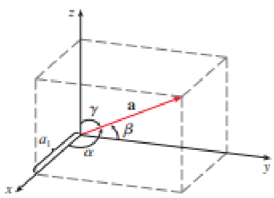
Dot product:

DEFINITION: $\mathbf{a} = \langle a_1, a_2, a_3 \rangle \& \mathbf{b} = \langle b_1, b_2, b_3 \rangle$ then DOT PRODUCT
(-) $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
(-) $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1b_1 + a_2b + 2$
SCALER PRODUCT (OR INNER PRODUCT) other name dot product because $\mathbf{a} \cdot \mathbf{b} \in \mathbb{R}$
Properties dot product: $\mathbf{a}.\mathbf{b} \& \mathbf{c} \in V_3$ and α scaler then:
(1) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ (2) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (3) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
(4) $(\alpha\mathbf{a}) \cdot \mathbf{b} = \alpha(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\alpha\mathbf{b})$ (5) $\mathbf{0} \cdot \mathbf{a} = 0$
ANGLE θ BETWEEN THE VECTORS $\partial \& \mathbf{b}$ starts at the origin where $0 \leq \theta \leq \pi$, if $\mathbf{a} \& \mathbf{b}$ parallel then $\theta = 0$ or $\theta = \pi$
Theorem: θ angle between vectors $\mathbf{a} \& \mathbf{b}$ then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$
PROOF:



$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos(\theta)$
Because $|OA| = |\mathbf{a}|, |OB| = |\mathbf{b}|$ and $|AB| = |\mathbf{a} - \mathbf{b}|$
$\Rightarrow |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos(\theta)$
using the given properties, we can conclude the theorem.
Corollary: $\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
PERPENDICALOR OR ORTHOGONAL: if angle between the vectors is $\theta - \frac{\pi}{2}$ so when $\mathbf{a} \cdot \mathbf{b} = 0$

Direction angles and direction cosines:

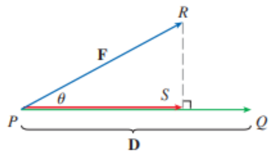


DIRECTION ANGLES: α, β, γ in above figure. (angle that $\mathbf{a}$ makes with the positive $x-, y-, z-$ axes.)
DIRECTION COSINES: the cosine of the direction angles:
(-) $\cos(\alpha) = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$
(-) $\cos\beta = \frac{a_2}{|\mathbf{a}|}$
(-) $\cos\gamma = \frac{a_3}{|\mathbf{a}|}$
Bu squaring we see that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ so $\mathbf{a} = \langle |\mathbf{a}| \cos\alpha, |\mathbf{a}| \cos\beta, |\mathbf{a}| \cos\gamma \rangle = |\mathbf{a}| \langle \cos\alpha, \cos\beta, \cos\gamma \}$
so $\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

Projections:

SCALER PROJECTION OF VECTOR B ONTO VECTOR A: $\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$
VECTOR PROJECTION OF VECTOR B ONTO VECTOR A: $\text{comp}_{\mathbf{a}}\mathbf{b} = (\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|})\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

Applications:



CONCSTANT FORCE VECTOR: $\mathbf{F}$
DISPLACEMENT VECTOR: $\mathbf{D}$
WORK: product of component of hte force along $\mathbf{D}$ and the distance moved.
$\mathbf{W} = (|\mathbf{F}|\cos(\theta)|\mathbf{D}| = |\mathbf{F}||\mathbf{D}|\cos(\theta) = \mathbf{F}\cdot\mathbf{D}$

Cross product:

CROSS PRODUCT: $\mathbf{a} = \langle a_1, a_2, a_3\rangle \& \mathbf{b} = \langle b_1, b_2, b_3\rangle$, then $\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$
Only when $\mathbf{a}\&\mathbf{b}$ three dimensional vectors.
DETERMINANT ORDER 2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
DETERMINANT OF ORDER 3: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$
So if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then we can say that:
$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
Orthogonal: The vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}\&\mathbf{b}$
PROOF:
Just 1 part:
$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\cdot a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\cdot a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\cdot a_3 = a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_2(a_1b_2 - a_2b_1) = 0$ so orthogonal.
angle between vectors and cross product: $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$
PROOF:
$|\mathbf{a}\times\mathbf{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 = (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3 +^2$
$= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta$
$= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2(\theta)$
$|\mathbf{a}|^2|\mathbf{b}|^2\sin^2(\theta)$
Take the square root of both sides and you see the result like in the theorem.
PARALLEL: $\mathbf{a}\times\mathbf{b} = \mathbf{0}$
LENGTH CROSS PRODUCT $\mathbf{a}\times\mathbf{b}$ equal to the area determined by $\mathbf{a}\&\mathbf{b}$

Algebra cross products:

For the standard basis vectors:
$\mathbf{i}\times\mathbf{j} = \mathbf{k}$ $\mathbf{j}\times\mathbf{k} = \mathbf{i}$ $\mathbf{k}\times\mathbf{i} = \mathbf{j}$
$\mathbf{j}\times\mathbf{i} = -\mathbf{k}$ $\mathbf{k}\times\mathbf{j} = -\mathbf{i}$ $\mathbf{i}\times\mathbf{k} = -\mathbf{j}$

For $\mathbf{a}, \mathbf{b}, \mathbf{c}$ vectors and scaler α:

(1) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (2) $(\alpha\mathbf{a}) \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\alpha\mathbf{b})$

(3) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (4) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

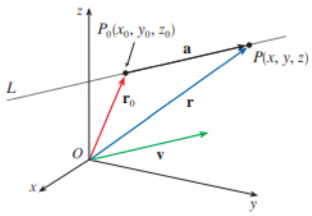
(5) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ (6) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Triple product:

Triple product: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Volume parallelepiped: determined by $\mathbf{a}, \mathbf{b}, \mathbf{c}$: $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}|$

Lines:



Triangle law for vector addition: $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$

Since $\mathbf{a}$&$\mathbf{v}$ parallel, exists scaler t s.t. $\mathbf{a} = t\mathbf{v}$ so: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

Where this last equation is called Vector equation of L

Parameter: t gives position vector $\mathbf{r}$

$\mathbf{r}$ can also be written as $\mathbf{r} = \langle x, y, z\rangle$

When $t\mathbf{v} = \langle ta, tb, tc\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ then: $\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$

Parametric equations:

(-) $x = x_0 + at$

(-) $y = y_0 + bt$

(-) $z = z_0 + ct$

where $t \in \mathbb{R}$ and L through $P(x_0, y_0, z_0)$ and parallel to $\langle a, b, c\rangle$

Each value of t gives a point on L

a, b, c are called direction numbers of L

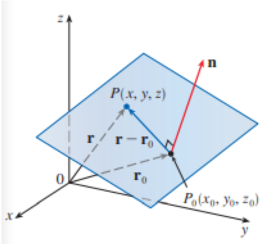
Summetric equations: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

Line segement from $\mathbf{r}_0$ to $\mathbf{r}_1$ given by:

$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$ where $0 \le t \le 1$

Skew lines: lines that doe no intersect.

Planes:



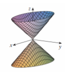
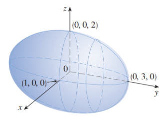
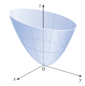
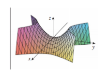
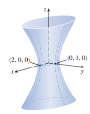
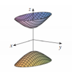
NORMAL VECTOR $\mathbf{n}$ orthogonal to the plane.
Let $P(x, y, z)$ arbitrary plane and $\mathbf{r}_0, \mathbf{r}$ position vectors of P_0 and P then $\mathbf{r} - \mathbf{r}_0 = \overrightarrow{P_0P}$
We see then that $n \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \Leftrightarrow n \cdot \mathbf{r} = n \cdot \mathbf{r}_0$
These equations are calleed the vector equation of the plane.
SCALER EQUATION OF THE PLANE trough $P_0(x_0, y_0, z_0)$ with $\mathbf{n} = \langle a, b, c\rangle$ is: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
Then we can write this plane to: $ax + by + cz + d = 0$
Where LINEAR EQUATION IN x, y, z: $d = -(ax_0 + by_0 + cz_0)$

DISTANCE D FROM THE POINT $P_1(x_1, y_1, z_1)$ TO THE PLAINE $ax+by+cz+d = 0$: $D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

Lecture 2

3 dimensional planes:

TRACES: curves intersection surface with planes ⊥ coordinate plane. RULLINGS: lines in a surface
QUADRIC SURFACE: second degree equations in 3 variables x, y, z and with constants: $A, \dots, J$
General form: $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$
Standard form 1: $Ax^2 + By^2 + Cz^2 + J = 0$ Standard form 2: $Ax^2 + By^2 + Cz + j = 0$

Name	Definition	Formula	Image
CYLINDER	surface that consist rullings Parallel given line,through a given plane		
CONE	Horizontal traces ellipses Vertical traces $x = k$ and $y = k$ hyperbolas if $k \neq 0$ otherwise pairs of lines	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	
PARABOLIC CYLINDER	made of inf. many shifted copies parabola		
ELLIPSOID	Traces are ellipses	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $a = b = c$? Sphere	
ELLIPTIC PARABOLOID	Horizontal traces ellipses Verticle traces parabolas	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ variable to first power indicate axis parabaloid	
HYPERBOLIC PARABOLOID	Horizontal traces parabolas Vertical traces parabolas	$\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}$	case where $c < 0$ 
HYPERBOLOID OF ONE SHEET:	Horizontal traces ellipses Vertical traces hyperbolas negative variable is axis symmetry	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	
HYPERBOLOID OF TWO SHEETS	Horizontal in $z = k$ ellipses if $k > c$ or $k < -c$ Vertical traces hyperbolas two minus signs: two sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	

Vector functions:

VECTOR FUNCTIONS: maps $\mathbb{R}$ to $\mathbb{R}^n$
COMPONENT FUNCTIONS: $I \subset \mathbb{R}$ and : $I \to \mathbb{R}^n$ and $t \to \langle r_1(t), \ldots, r_n(t) \rangle$
Example: $n = 3$ then $r(t) = \langle g(t), h(t), k(t) \rangle$
Definition 1:
If $\mathbf{r}(t) = \langle f(t0 < g(t), H(t) \rangle$ then $\lim_{t\to a} \mathbf{r}(t) =\rangle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$
Provides, limits of component functions exists.

PROOF:
recall $f(t) = f_1(t), g(t) = f_2(t)$ and $h(t) = f_3(t)$
$0 < |t - a| < \Delta \Rightarrow \|\mathbf{r}(t) - L\| < \varepsilon$
$\exists \delta_i > 0$ s.t. $0 < |t - a| < \delta_i \Rightarrow |f_i(t) - L_i| < \frac{\varepsilon}{\sqrt{3}}$ for $i = 1, 2, 3$

Set $\delta = \min\{\delta_i\}$ so then $\|\mathbf{r}(t) - L\| = \sqrt{\sum_{i=1}^{3} (f_i(t) - L_i)^2} \leq \sqrt{\frac{\varepsilon^2}{3} + \frac{\varepsilon^2}{3} + \frac{\varepsilon^2}{3}} = \varepsilon$

Distance vectors: $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ defined by $\|\mathbf{u} - \mathbf{v}\| = \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2}$
CONTINUOUS: $\mathbf{r} : I \to \mathbb{R}^n$ continuous at $a \in I$ if $\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$
SPACE CURVE: $C = \mathbf{r}(I)$ where $I \subset \mathbb{R}$ interval and $\mathbf{r} : I \to \mathbb{R}^3$ where $\mathbf{r}$ the PARIMACTERISATION OF C
New spaces in this chapter without explanations:
Helix, toroidal spiral (lies on torus), trefoil knot, twisted cube

Lecture 3:

Definition 1:
DERIVATIVE $\mathbf{r}'(t)$ defined as $\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$
Remarks:
□ $\mathbf{r}'(t) =$ tangent vector of the curve $C = \mathbf{r}(I)$ at the point $\mathbf{r}(t)$ where $t \in I$
□ UNIT TANGENT VECTOR $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ as long as $\mathbf{r}'(t) \neq 0$
Theorem 2:
If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ where f, g, h differentiable:
$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$
Remarks:
□ second derivative also possible: $\mathbf{r}"(t) = (\mathbf{r}'(t))'$
Theorem 3:
$\mathbf{u}, \mathbf{v}$ are vectors, c is a scaler and f real valued function:

1	$\frac{d}{dt}$	$[\mathbf{u}(t) + \mathbf{v}(t)]$	$= \mathbf{u}'(t) + \mathbf{v}'(t)$
2	$\frac{d}{dt}$	$[c\mathbf{u}(t)]$	$= c\mathbf{u}'(t)$
3	$\frac{d}{dt}$	$[f(t)\mathbf{u}(t)]$	$= f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4	$\frac{d}{dt}$	$[\mathbf{u}(t)\cdot\mathbf{v}(t)]$	$= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
5	$\frac{d}{dt}$	$[\mathbf{u}(t)\times\mathbf{v}(t)]$	$= \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
6	$\frac{d}{dt}$	$[\mathbf{u}(f(t))]$	$= f'(t)\mathbf{u}'(f(t))$

Integrability,arclength and reparemeterization:

INTEGRABILITY: vector function integrable on interval $I \Leftrightarrow$ components integrable on I
$\int_a^b \mathbf{r}(t)dt = (\int_a^b f(t)dt)\mathbf{i} + (\int_a^b g(t)dt)\mathbf{j} + (\int_a^b h(t)dt)\mathbf{k}$
$I = [a, b]$ and $\mathbf{r} : I \to \mathbb{R}^3$ continous differentiable s.t. $\mathbf{r}'(t)$ exists. Then $\mathbf{r}$ is of class C^1

We know that the length of a vector function S_i is given by: $\Delta S_i = \|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\|$
Where $\Delta x_i = f(t_i) - f(t_{i-1})$ and $\Delta y_i = g(t_i) - g(t_{i-1})$ and $\Delta z_i = h(t_i) - h(t_{i-1})$
So $\Delta S_i = \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2}$
ARCLENGTH OF $C' = \mathbf{r}(I)$: $\lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} \Delta S_i$

Theorem 1	$\mathbb{R}^2$	$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt$	$= \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$
Theorem 2	$\mathbb{R}^3$	$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt$	$= \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt$

We can rewrite this all to $L = \int_a^b |\mathbf{r}'(t)|dt$ **Theorem 3**

If $\mathbf{r}(t) = f(t)\mathbf{i}(t) + g(t)\mathbf{j} + h(t)\mathbf{k}$ where $a \leq t \leq b$ and $\mathbf{r}(t)$ is at least of class C^1 then:
Theorem 6,7: ARC LENGTH FUNCTION: $s(t) = \int_a^t |\mathbf{r}'(u)|du = \int_a^t \sqrt{(\frac{dx}{du})^2 + (\frac{dy}{du})^2 + (\frac{dz}{du})^2}du$ so then we see that $\frac{ds}{dt} = |\mathbf{r}'(t)|$
PARAMETERIZE A CURVE W.R.T. ITS ARC LENGTH: usefull method. Set the arc length equal to a function $s(t)$ and subsitute $t = s(t)$ in the original vector function.

Example:

A single curve can be represented by more then 1 vector function. For example:
theorem 4: (1) $\mathbf{r}_1(t) = \langle t, t^2, t_3 \rangle$ where $1 \le t \le 2$
theorem 5: (2) $\mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle$ where $0 \le u \le \ln(2)$
Gives exactly the same graph

Independent length:
Lenght of curve C' does not depend on the parameterization in the following sense:
$\int\limits_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int\limits_c^d \left\| \frac{d\tilde{\mathbf{r}}}{du} \right\| du$
$h : [a, b] \to [c, d] C'$ and bijective. so $t \to u = h(t)$ s.t. $\mathbf{r}(t) = \tilde{\mathbf{r}}(h(t))$
PROOF:
recall substitution rule integrals. $\int\limits_a^b f(g(x))g'(x)dx = \int\limits_{g(a)}^{g(b)} f(u)du$

$$\int\limits_a^b \left\| \frac{dr}{dt} \right\| dt = \int\limits_a^b \left\| \frac{d\tilde{r}(h(t))}{dt} \right\| dt = \int\limits_a^b \|\tilde{r}'(h(t)) \cdot h'(t)\| dt$$

$$= \int\limits_a^b \|\tilde{r}'(h(t))\| |h'(t)| dt = \begin{cases} \int\limits_a^b \|\tilde{r}'(h(t))\| |h'(t)| dt, h' \ge 0 \\ -\int\limits_a^b \|\tilde{r}'(h(t))\| |h'(t)| dt, h' < 0 \end{cases} = \int_c^d \left\| \frac{d\tilde{r}}{du}(u) \right\| du = du$$

Because when first case $a \to c$ and $b \to d$ so then $\mathbf{r} = \tilde{\mathbf{r}}$
Second case $a \to d$ and $b \to c$ so then $\mathbf{r} \to -\tilde{\mathbf{r}}$
Note:
One natural parameterization of a curve is parameterization by arclength: $s(t) = \int\limits_a^t \|\mathbf{r}'(t)\| dt =$ length of the position of the curve c between the points $\mathbf{r}(a)$ and $\mathbf{r}(t)$
$s(t)$ resp. corresponds to $h(t)$ resp. to u in proposition above. Then $c = 0$ and $d = L$
Remarks:
$\frac{ds}{dt} = \|\mathbf{r}'(t)\|$
in phyiscs: $\frac{ds}{dt}$ corresponds to the norm of the velocity vector, which we call speed.

Lecture 4:

Curvature:

SMOOTH CURVE if the curve has a SMOOTH PARAMETERIZATION: $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$
Recall: Unit tangent: Indicates direction of curve: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
Definition 8: CURVATURE: The rate of change of unit tangent vector w.r.t. arc length. curve of class C^2 where $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$
Theorem 9 and 10 when we substitute $\frac{ds}{dt} = |\mathbf{r}'(t)|$ and after that fill in the formula for the unit tangent vector we find $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$
Theorem 11: when we have the curvature $y = f(x)$ then $\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{1.5}}$

Moving frames and torsion:

Let $C : \mathbf{r} : I \to \mathbb{R}^3$ of class C^3 then we can find 4 mutually orthogonal vectors of length 1 at each point of C
UNIT TANGENT VECTOR: $\mathbf{T}(r) = \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t)$
(PRINCIPAL) UNIT NORMAL (VECTOR): direction in which the curve is turning at each point. $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$
BINORMAL VECTOR: perpendicular to $\mathbf{T}$ and $\mathbf{N}$ defined by $\mathbf{B}(t) = \mathbf{N}(t) \times \mathbf{T}(t)$
NORMAL PLANE: the blane determine by $\mathbf{N}$ and $\mathbf{B}$ at a point P on a curve C
OSCULATING PLANE: The plane determined by $\mathbf{T}$ and $\mathbf{N}$ of C at a point P
OSCULATING CIRCLE/CIRCLE OF CURVATURE: circle lies in oscolating plane, same tangetn at C at P on the side on towards $\mathbf{N}$ points, and has radius $\rho = \frac{1}{\kappa}$
TORSION: (τ) which we can find by **Definition 13** $\tau = -\frac{d\mathbf{B}}{ds}\mathbf{N} = -\tau\mathbf{N}$ measures how spatial (non planair) a curve is.
,or **Definition 12:** $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$
Definition 14: $\tau(t) = -\frac{\mathbf{B}'(t)\cdot\mathbf{N}(t)}{|\mathbf{r}'(t)|}$
It can be shown that: $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$ and $\frac{d\mathbf{B}}{ds} = -\tau N$ but $\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$
So $\begin{pmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}'\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}$ which is called the Frenet-serret equations.
TORSION OF A CURVE BY THE VECTOR FUNCTION: **Theorem 15:** $\tau(t) = \frac{[\mathbf{r}'(t)\times\mathbf{r}''(t)]\cdot\mathbf{r}'''(t)}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|^2}$

Example:

$\mathbf{r} : [-1, 1] \to \mathbb{R}^2$ so $t \to \langle t^3, t^2\rangle$ so $y = x$ gives $t^2 = t^3$ so $t = \sqrt{t^3}$

$\mathbf{r}(t) = a\cos(t)\mathbf{i} + a\sin(t)\mathbf{j} + bt\mathbf{k}$ where $a, b \geq 0$
$\mathbf{T}(t) = \frac{1}{\|\mathbf{r}'(t)\|}\mathbf{r}'(t) = \frac{-a\sin(t)\mathbf{i}+a\cos(t)\mathbf{j}+bk}{\sqrt{a^2+b^2}}$
$\mathbf{N}(t) = \frac{1}{\|\mathbf{T}'(t)\|}\mathbf{T}'(t) = \frac{\frac{-a\cos(t)\mathbf{i}-a\sin(t)\mathbf{j}}{\sqrt{a^2+b^2}}}{\frac{a}{\sqrt{a^2+b^2}}} = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j}$
$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{a}{a^2+b^2}$
The curvuture of a circle is given by $\frac{1}{r}$ where $r =$ radius.
$\mathbf{B} = \mathbf{T} \times \mathbf{N} = (\frac{b}{\sqrt{a^2+b^2}}\sin(t))\mathbf{i} - (\frac{b}{\sqrt{a^2+b^2}}\cos(t))\mathbf{j} + \frac{a}{\sqrt{a^2+b^2}}\mathbf{k}$

Note: $\frac{d\mathbf{B}}{dt} = (\frac{b}{\sqrt{a^2+b^2}}\cos(t))\mathbf{i} + (\frac{b}{\sqrt{a^2+b^2}}\sin(t))\mathbf{j}$
So we see that this vector is parallel to $\mathbf{N}$

Application: linear approximation:

$\mathbf{r} : I \subset \mathbb{R} \to \mathbb{R}^n$ different at $t \in I$ so:
$\exists \mathbf{v} \in \mathbb{R}^n$ s.t. $\lim\limits_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} = \mathbf{v}$
$\Leftrightarrow \exists \mathbf{v} \in \mathbb{R}^n$ s.t. $\lim\limits_{\tau\to 0} \frac{\mathbf{r}(\tau)-\mathbf{r}(t)}{\tau - t} = \mathbf{v}$
$\Leftrightarrow \exists \mathbf{v} \in \mathbb{R}^n$ s.t. $\lim\limits_{\tau\to t} \frac{\mathbf{r}(\tau)-(\mathbf{r}(t)+\mathbf{v}(\tau-t))}{\tau - t} = 0$
$\Leftrightarrow \mathbf{r}(t) + \mathbf{v}(\tau - t)$ the linear approximation of the function $\mathbf{r}$ at $\mathbf{r}(t)$
$L(\tau) = \mathbf{r}(t) + \mathbf{v}(\tau - t)$ so the linearisation of $\mathbf{r}$

Lecture 5:

functions:

Definition let $(x,y) \to f(x,y)$ Then:
DOMAIN: $(x,y) \in D$ then D domain.
RANGE: $\{f)x,y)|(x,y) \in D\}$
When we have $z = f(x,y)$ then x, y INDEPENDENT VARIABLES and z DEPENDENT VARIABLES.
GRAPH: if f function two variables with domain D then GRAPH set of all points $(x,y,z) \in \mathbb{R}^3$ s.t. $z = f(x,y)$ and $(x,y) \in D$
LEVEL CURVES: f two variables are the curves with equations $f(x,y) = k$ where k constant in range f
CONTOUR/LEVEL MAP: collection of level curves.
FUNCTION OF 3 VARIABLES: ordered triple $(x,y,z) \in D \subset \mathbb{R}^3$ where D domain assings to a unique real number $f(x,y,z)$
HALF-SPACE CONSISTING ALL POINTS ABOVE PLANE, $z = y$: $D = \{(x,y,z) \in \mathbb{R}^3 | z > y\}$
LEVEL SURFACES: surfaces s.t. $f(x,y,z) = k$ where k a constant.

Example:

A company uses n different ingedients in making a food product, where c_i is the cost per unit of the ith ingredient, you need x_i units of the ith ingredient, then the total cost:
$C = f(x_1, \ldots, x_n) = c_1x_1 + \ldots + c_nx_n$
We can rewrite this to $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$
There are three ways of looking at a function f defined on subset $\mathbb{R}^n$:
(1) function real variables $x_1, \ldots, x_n$ (2) function single point variable $(x_1, \ldots, x_n)$
(3) function single vector variable $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$

Limits and continuous

Definition 1: f function 2 variables, domain D includes points arbitrarily close to (a,b). Then LIMIT OF $f(x,y)$ AS $(x,y) \to (a,b)$ IS L: if for every $\varepsilon > 0$ there $\exists \delta > 0$ s.t.:
if $(x,y) \in D$ and $9 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon$
Notation: $\lim\limits_{(x,y)\to(a,b)} f(x,y) = \lim\limits_{\substack{x\to a\\ y\to b}} = L$ and $f(x,y) \to L$ as $(x,y) \to (a,b)$
Existence of a limit:
If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2 where $L_1 \neq L_2$ then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example:

1:
$f : \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \to 3x - 5y$ show $\lim\limits_{(x,y)\to(1,-1)} f(x,y) = 8$
Let $\varepsilon > 0$ to be shown,;$\exists \delta > 0$ s.t. $0 < \|(x,y) - (1,-1)\| < \delta$ implies $|3x - 5y - 8| < \varepsilon$
$\left.\begin{matrix}|x-1|\\|y+1|\end{matrix}\right\} \leq \|(x,y) - (1,-1)\| = \sqrt{(x-1)^2 + (y+1)^2} < \delta$ it follows that $|3x - 5y - 8| = |3(x+1) - 5(y+1)| \leq$
$|3(x-1)| + |-5(y+1)| = 3|x-1| + 5|y+1|$
We know that $|x-1| < \delta$ and $|y+1| < \delta$

So we see that $\|(x, y) - (1, -1)\| \leq 8\delta$ so then we can set $\varepsilon = \frac{\delta}{8}$ so then we see that $\|(x - y) - (1, -1)\| < \varepsilon$

2:

$f : \mathbb{R}^2 \setminus \{(0, 0\} \to \mathbb{R}$

$(x, y) \to f(x, y) = \frac{x^2-y^2}{x^2+y^2}$ does this function have a limit at $(x, y) = (0, 0)$?

$f(x, 0) = \frac{x^2}{x^2} = 1$ true for all $x \neq 0$

$f(0, y) = \frac{-y^2}{y^2} = -1$ for all $y \neq 0$

f has no limit at the the point $(x, y) = (0, 0)$

3:

Sometimes polar coordinates useful to decide whether function has limit.

$x = r\cos(\theta)$ and $y = r\sin(\theta)$

does $f(x, y) = \frac{x^3+x^5}{x^2+y^2}$ have a limit at the origin?

$\frac{x^3+x^5}{x^2+y^2} = \frac{r^3\cos^3(\theta)+r^5\cos^5(\theta)}{r^2\cos^2(\theta)+r^2\sin^2(\theta)} = r(\cos^3(\theta) + r^2\cos^5(\theta)) = r\cos(\theta)(\cos^2(\theta) + r^2\cos^4(\theta))$

Because $|\cos(\theta)| \leq 1$ for all θ

Hence:

$-r(1 + r^2) \leq r\cos(\theta)(\cos^2(\theta) + r^2\cos^4(\theta)) \leq r(1 + r^2)$

When $x, y \to 0$ we know that $r \to 0$ and therefore $-r(1 + r^2) \to 0$ and $r(1 + r^2) \to 0$ so by squeezing theorem: $\lim_{(x,y)\to(0,0)} f(x, y) \to 0$

Properties of limits:

Sum Law	$\lim[f(x) + g(x)] = \lim f(x) + \lim g(x)$
Differnece law	$\lim[f(x) - g(x)] = \lim f(x) - \lim g(x)$
Constant multiple	$\lim[cf(x)] = c\lim f(x)$
Product law	$\lim[f(x)g(x)] = \lim f(x)\lim g(x)$
Quotient rule	$\lim[\frac{f(x)}{g(x)}] = \frac{\lim f(x)}{\lim g(x)}$ where $\lim g(x) \neq 0$
2(& below)	$\lim_{(x,y)\to(a,b)} x = a$
	$\lim_{(x,y)\to(a,b)} y = b$
	$\lim_{(x,y)\to(a,b)} c = c$

Polynomial fucntion: sum of terms of the form cx^my^n where c constant and $m, n \geq 0$

Rational function: ratio two polynomials.

Definition 3: $\lim_{(x,y)\to(a,b)} p(x, y) = p(a, b)$

Definition 4: $\lim_{(x,y)\to(a,b)} q(x, y) = \lim_{(x,y)\to(a,b)} \frac{p(x,y)}{r(x,y)} = \frac{p(a,b)}{r(a,b)} = q(a, b)$

Definition 6: f continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$. Continuous on domain D if it is continuous at every $(a, b) \in D$

Definition 7: f defined on subset D of $\mathbb{R}^n$ then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means:

$\forall\varepsilon > 0\exists\delta > 0$ s.t. $\mathbf{x} \in D$ and $0 < |\mathbf{x} - \mathbf{a}| < \delta$ then $|f(\mathbf{x}) - L| < \varepsilon$

Continuity of a vector:

$\mathbf{a} \in D$ and $\lim_{\mathbf{x}\to\mathbf{a}} f(x) = f(a)$ then f continuous at a

Derivatives of functions:

Definition 4:

Definition 1 and 2: PARTIAL DERIVATIVE OF F W.R.T. X $f_x(a,b) = g'(a)$ where $g(x) = f(x,b)$ so $f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$

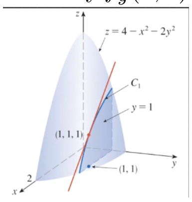
Definition 3: PARTIAL DERIVATIVE OF F W.R.T. Y ,$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$

Notation:
$f_x(x,y) = f_x = \frac{\delta f}{\delta x} = \frac{\delta}{\delta x} f(x,y) = \frac{\delta z}{\delta x} = f_1 = D_1 f = D_x f$
$f_y(x,y) = f_y = \frac{\delta f}{\delta y} = \frac{\delta}{\delta y} f(x,y) = \frac{\delta z}{\delta y} = f_2 = D_2 f = D_y f$
Rules:
To find f_x regard y constante, differentiate $f(x,y)$ w.r.t. x Finding f_y similar.

If $u = f(x_1,\ldots,x_n)$ then $\frac{\delta u}{\delta x_i} = \lim_{h\to 0} \frac{f(x_1,\ldots,x_{i-1},x_i+h,\ldots,x_n)-f(x_1,\ldots,x_n)}{h} = \frac{\delta f}{\delta x_i} = f_{x_i} = f_i = D_i f$

Example:

$D \subset \mathbb{R}^2$ where $f(x,y) = 4 - x^2 - 2y^2$
$f_x(1,1) = \lim_{h\to 0} \frac{f(1+h,1)-f(1,1)}{h} = \lim_{h\to 0} \frac{-2h-h^2}{h} = \lim_{h\to 0} -2-h = -2$
Similary $f_y(1,1) = -4$



Cruve C_1' parameterization: $r_1 = x \to (x, 1, f(x,1)) = (x, 1, 4 - x^-2) = (x, 1, 2 - x^2)$

Higher derivatives:

We can also compute the second partial derivative:
$(f_x)_x = f_{xx} = f_{11} = \frac{\delta}{\delta x}(\frac{\delta f}{\delta x}) = \frac{\delta^2 f}{\delta x^2} = \frac{\delta^2 z}{\delta x^2}$
$(f_x)_y = f_{xy} = f_{12} = \frac{\delta}{\delta y}(\frac{\delta f}{\delta x}) = \frac{\delta^2 f}{\delta x \delta y} = \frac{\delta^2 z}{\delta x \delta y}$
$(f_y)_y = f_{yy} = f_{22} = \frac{\delta}{\delta y}(\frac{\delta f}{\delta y}) = \frac{\delta^2 f}{\delta y^2} = \frac{\delta^2 z}{\delta y^2}$
$(f_y)_x = f_{yx} = f_{21} = \frac{\delta}{\delta x}(\frac{\delta f}{\delta y}) = \frac{\delta^2 f}{\delta y \delta x} = \frac{\delta^2 z}{\delta y \delta x}$
Clairaut's theorem: Suppose f defined on disk D that contains (a,b). If f_{xy} and f_{yx} both continuous on D then $f_{xy}(a,b) = f_{yx}(a,b)$
HARMONIC FUNCTIONS: solution of the LAPLACE'S EQUATION: $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$
WAVE EQUATION: $\frac{\delta^2 u}{\delta t^2} = a^2 \frac{\delta^2 u}{\delta x^2}$ decribes motion of waveform.

Tangent plane,linear approximation:

Definition 2: f continuous partial derivative. Then equation tangent plane surface $z = f(x,y)$ at $P(x_0,y_0,z_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$
LINEARIZATION: **Definition 3:** $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Linear approximation or tangent plane approximation:

$\mathbb{R}^2$ **Definition 4:** $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$\mathbb{R}^3$ $f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$

Lecture 6:

Differentiability:

Theorem 5: f differentiable at a then $\Delta y = f'(a)\Delta x + \varepsilon \Delta x$ where $\varepsilon \to 0$ as $\Delta x \to 0$

INCREMENT: change in value of f when (x, y) changes from (a, b) to $(a+\Delta x, b+\Delta y)$: $\mathbb{R}^2$ textbfDefinition 6: Δz =
$\mathbb{R}^3$ $\Delta w = f(x$

DIFFERENTIABLE:
(1) **Definition 7:** If $z = f(x, y)$ then f differentiable at (a, b) if:
$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$
When $(\Delta x, \Delta y) \to (0, 0)$ then $\varepsilon_1 \& \varepsilon_2 \to 0$
(2) **Theorem 8:** if partial derivatives f_x and f_y exists near (a, b) and continuous at (a, b) then f differentiable at (a, b)

Differentials:

We already now that the differential of y is defined as $dy = f'(x)dx$ when $y = f(x)$ **Definition 9.**
TOTAL DIFFERENTIAL
$\mathbb{R}^2$ **Definition 10:** $dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\delta z}{\delta x}dx + \frac{\delta z}{\delta y}dy$
$\mathbb{R}^3$ $dw = \frac{\delta w}{\delta x}dx + \frac{\delta w}{\delta y}dy + \frac{\delta w}{\delta z}dz$

Chain rule:

Theorem	Cpnditions	The chain rule	etc.
Theorem 1:	$z = f(x, y)$,$x = g(t)$,$y = h(t)$	$\frac{dz}{dt} = \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt}$	equal to $\frac{dz}{dt} = \frac{\delta z}{\delta x}\frac{dx}{dt} + \frac{\delta z}{\delta y}\frac{dy}{dt}$
Theorem 2:	$z = f(x, y)$,$x = g(s, t)$ and $y = h(s, t)$	$\frac{\delta z}{\delta s} = \frac{\delta z}{\delta x}\frac{\delta x}{\delta s} + \frac{\delta z}{\delta y}\frac{\delta y}{\delta s}$ $\frac{\delta z}{\delta t} = \frac{\delta z}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta z}{\delta y}\frac{\delta y}{\delta t}$	INDEP.VAR.: $s \& t$ INTER. VAR.: x, y DEP. VAR.: z
Theorem 3	$u = (x_1, \ldots, x_n)$	$\frac{\delta u}{\delta t_i} = \frac{\delta u}{\delta x_1}\frac{\delta x_1}{\delta t_i}$ $+ \ldots + \frac{\delta u}{\delta x_n}\frac{\delta x_n}{\delta t_i}$	each x_j differentiable on $t_1, \ldots, t_m$

Implicit Function theorem:

Theorem 5:
$\frac{dy}{dx} = -\frac{\frac{\delta F}{\delta x}}{\frac{\delta F}{\delta y}} = -\frac{F_x}{F_y}$
CONDITIONS:
(1) F defined on a disk containing (a, b)
(2) $F(a, b) = 0$,but $F_y(a, b) \neq 0$
(3) F_x and F_y continuous on disk.
$\Rightarrow$ then $F(x, y) = 0$ deifnes y as function of x near (a, b) derivative given by function above.

Theorem 6: similar to 5:
$\frac{\delta z}{\delta x} = -\frac{\frac{\delta F}{\delta x}}{\frac{\delta F}{\delta z}} - 0\frac{F_x}{F_y}$ and $\frac{\delta z}{\delta y} = \frac{\frac{\delta F}{\delta y}}{\frac{\delta F}{\delta z}} = -\frac{F_y}{F_z}$
Where F on sphere containing (a, b, c) and $F(a, b, c) = 0$ and $F_z(a, b, c) \neq 0$ and F_x, F_y, F_z continuous inside sphere, then $F(x, y, z) = 0$ defines z as function x and y near (a, b, c) then function differentiable.

Lecture 7:

Direction derivative:

Two dimensional:

14.6:
Theorem 1:
$z = f(x, y)$ then we have:
$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$ and $f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$ partial derivatives.
DIRECTIONAL DERIVATIVES:
$f_x(x_0, y_0)$ is rate of change z in direction of x so the direction of unit vector $\mathbf{j}$ (similar for $f_y(x_0, y_0)$ and z)
Theorem 2: DIRECTION DERIVATIVE of f at (x_0, y_0) in the direction of unit vector $\mathbf{u} = \langle a, b\rangle$ is:
$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+ha,y_0+hb)-f(x_0,y_0)}{h}$ if this limit exists
Theorem 3: $D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$ where $\mathbf{u} = \langle a, b\rangle$ and $f_{\mathbf{u}}$ the directional derivative.
Definition 8GRADIENT: if f function 2 variables, then GRADIENT OF: f
$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\delta f}{\delta x}\mathbf{i} + \frac{\delta f}{\delta y}\mathbf{j}$
Rewriting 7:
$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle = \langle f_x(x, y), f_y(x, y)\rangle \cdot \mathbf{u}$
Definition 9: $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$

3 dimensional:

Theorem 10: DIRECTIONAL DERIVATIVES: f at (x_0, y_0, z_0) of $\mathbf{u} = \langle a, b, c\rangle$ is:
$D)\mathbf{u} f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0+ha,y_0+hb,z_)+hc)-f(x_0,y_0,z_0)}{h}$ if limit exists.
Theorem 11: $D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h\to 0} \frac{f(\mathbf{x}_0+h\mathbf{u})-f(\mathbf{x}_0)}{h}$
Theorem 12: $D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$
Theorem 13: GRADIENT: $\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\delta f}{\delta x}\mathbf{i} + \frac{\delta f}{\delta y}\mathbf{j} + \frac{\delta f}{\delta z}\mathbf{k}$
Theorem 14: $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$

maximize

Theorem 15: suppose f differentiable function 2 or 3 variables. Maximum value of $D_{\approxeq} f(\mathbf{x}) = |\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ same direction as $\nabla f(\mathbf{x})$

Example:

$f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x^2 + y^2$
So $\nabla f(x_0, y_0) = (2x_0, 2y_0)$
So the levels will be circles. When we draw the vectors, we see that the vector is perpendicular to the tangent line at the circle.

Tangent plane level surfaces:

Let S surface with equation $F(x, y, z) = k$. So level surface function F. Let $P(x_0, y_0, z_0)$ on S.
Let C any curves on S through P. Then $C : \mathbf{r}(t) = \langle x_0, y_0, z_0\rangle$. Let t_0 correspond to P so:
$\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$ but we can rewrite this to:
Statement 16: $F(x(t), y(t), z(t)) = k$ and when F differentiable then by chain rule:
Statement 17: $\frac{\delta F}{\delta x}\frac{dx}{dt} + \frac{\delta F}{\delta y}\frac{dy}{dt} + \frac{\delta F}{\delta z}\frac{dz}{dt} = 0$
But therefore **Statement 18:** $\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$
Theorem 19: TANGENT PLANE TO LEVEL SURFACES: if $\nabla F(x_0, y_0, z_0) \neq 0$ then the tangent plane is equal to: $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$
NORMAL LINE: to S at P is the line through P perpendicular to S given by:
Theorem 20: $\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$

Properties of gradient:

Let f differentiable and $\nabla f(\mathbf{x}) \neq \mathbf{0}$ then:
(1) DIRECTIONAL DERIVATIVE $D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$
(2) $\nabla f(\mathbf{x})$ points in direciton maximum rate increasing f at $\mathbf{x}$ and maximum rate $|\nabla f(\mathbf{x})|$
(3) $\nabla f(\mathbf{x})$ perpendicular to level curve or level surfaces of f through $\mathbf{x}$

maxima and minima:

14.7:
Definition 1: Function 2 variables then:
LOCAL MAXIMUM(MINIMUM) at (a, b) if $f(x, y) \leq (\geq) f(a, b)$ when (x, y) near (a, b)
So $f(x, y) \leq (\geq) f(a, b)$ for all points (x, y) in some disk with center (a, b).
LOCAL MAXIMUM (MINIMUM) VALUE name of $f(a, b)$ in this case.
Theorem 2: f local maximum or minimum at (a, b) and first order partial derivatives f exists at (a, b) then $f_x(a, b) = 0$ and $f_y(a, b) = 0$
CRITICAL POINT OR STATIONARY: of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or one of these partial derivatives does not exists.
So then $\nabla f(a, b) = 0$
SADDLE POINT: if $f_x(a, b) = f_y(a, b) = 0$ but $f(a, b)$ is not a local maximum and not a local minimum.

Example:

$D = \mathbb{R}^2$,then $f(x, y) = 1 - |x| - |y|$ then f global maximum at $(x, y) = (0, 0)$
1: $D = \mathbb{R}^2$ then $f(x, y) = \frac{1}{3}x^3 - x + y^2 = g(x) + h(y)$

Lecture 8:

maxima and minima continued:

$f : D \subset \mathbb{R}^2 \to \mathbb{R}$ of class C^2 and has critical point $(a, b) \in D$

$d = \det(\text{HESSIAN MATRIX}) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b) \cdot f_{yy}(a,b) - (f_{xy}(a,b))^2$

Case 1: $d > 0$ and $f_{xx}(a,b) > 0$ then f local minimum at (a,b)

Case 2: $d > 0$ and $f_{xx}(a,b) < 0$ then f local maximum at (a,b)

Case 3: $d < 0$ then f has a saddle at (a,b)

Theorem 7: Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ where $(a,b) \in D$ then $f(a,b)$ is a ABSOLUTE MAXIMUM(MINIMUM) if $f(a,b) \geq (\leq) f(x,y)$ for all $(x,y) \in D$

CLOSED SET: if a set contains its boundaries. the complement of this set is open.

BOUNDED SET: set that contains not all of its boundarys.

Theorem 8: extreme value theorem for two functions of two variables: if f continuous on closed& compact set $D \subset \mathbb{R}^n$ thne$f$ attains absolute maximum at $f(x_1, y_1)$ and absolute minimum $f(x_2, y_2)$ for $(x_1, y_1)\&(x_2, y_2) \in D$

Theorem 9: to find absolute maximum (minimum) on closed and bounded set:

(1) find $f(a,b)$ where (a,b) critical point in D

(2) find extreme values on boundaries

(3) the largest (smallest) value of step 1 and step 2 is the absolute maximum (minimum) value.

Lagrange multipliers

14.8:

Theorem 1: When $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ where $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$ there exists LAGRANGE MULTIPLIER λ s.t. $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$

PROOF:

$t \to \mathbf{r}(t)$ parameterization of a curve in S s.t. $\mathbf{r}(t) = a$

Then $(f \circ \mathbf{r})(t)$ extremum at t_0

Hence $\frac{d}{dt} f(\mathbf{r}(t_0)) = \nabla f(\mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0) = \nabla f(a) \cdot \mathbf{r}'(t_0) = 0$

This holds for all curves in S at $a \in S$

Together with the tangent vectors span tangent plane of S at $a \in S$

So $\nabla f(a) \perp S@a$ and hence is parallel to $\nabla g(a)$

Method lagrange multipliers:

Find maximum& minimum values $f(x,y,z)$ to the constraint $g(x,y,z) = k$ assuming extreme values exists, and $\nabla g \neq \mathbf{0}$ on $g(x,y,z) = k$

(1) find all values s.t. $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z) = k$

(a) $f_x(x,y,z) = \lambda g_x(x,y,z)$ and $f_y(x,y,z) = \lambda g_y(x,y,z)$ and $f_z(x,y,z) = \lambda g_z(x,y,z)$

(2) evaluate f at the founded values of (x,y,z) the largest: maximum value of f smallest: minimum value of f

Theorem 16: LAGRANGE MULTIPLIERS TWO CONSTRAINS:

$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$

So then $f_x = \lambda g_x + \mu h_x$ and $f_y = \lambda g_y + \mu h_y$ and $f_z = \lambda g_z + \mu h_z$

Furthermore $g(x,y,z) = k$ and $h(x,y,z) = c$

Lecture 9:

Double integral:

Definition 1: RIEMANNSUM: $\sum_{i=1}^{n} f(x_i^\star)\Delta x$ and **Definition 2:** INTEGRAL: $\int_a^b f(x)dx = \lim_{n\to\infty} f(x_i^\star)\Delta x$

SAMPLE POINT $(x_{ij}^\star, y_{ij}^\star)$ in each R_{ij}

Definition 3: So then we have that $V = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^\star, y_{ij}^\star)\Delta A$

VOLUME of the solid S that lies under f and above rectangle R

Definition 4: $V = \lim_{(m,n)\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^\star y_{ij}^\star)\Delta A$

Definition 5: DOUBLE INTEGRAL of f over rectangle R is:

$\int\int_R f(x,y)dA = \lim_{(m,n)\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^\star y_{ij}^\star)\Delta A$

If this limit exists.

f is INTEGRABLE if the limit in definition 5 exists.

DOUBLE RIEMANN SUM: the double sum in definition 5.

Definition 6:

If we choose $(x_{ij}^\star, y_{ij}^\star) = (x_i, y_i)$ then we get:

$\int\int_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{i=1}^{n} f(x_i, y_i)\Delta A$

So therefore, if $f(x,y) \geq 0$ then V volume lies above rectangle R and below surface $z = f(x,y)$ is $V = \int\int_R f(x,y)dA$

Midpoint rule:

$\int\int_R f(X,y)dA = \sum_{i=1}^{m}\sum_{i=1}^{n} f(\overline{x_i}, \overline{y_i})\Delta A$ where $\overline{x_i}$ midpoint $[x_{i-1}, x_i]$ and $\overline{y_i}$ midpoint $[y_{i-1}, y_i]$

Iterated integarls:

Suppose f integrable function on $R = [a,b] \times [c,d]$

PARTIAL INTEGRATION W.R.T. Y: held the other variables fixed and integrate with respect ot y

We see that $A(x) = \int_c^d f(x,y)dy$

Definition 7: $\int_a^b A(x)dx = \int_a^b [\int_c^d f(x,y)dy]dx$

ITERATED INTEGRAL: The integral on the right side.

Theorem 10: Fubini's theorem: f continuous on rectangle: $R = \{(x,y)|a \leq x \leq b, c \leq y \leq d\}$ then:

$\int\int_R f(x,y)dA = \int_a^b\int_c^d f(x,y)dydx = \int_c^d\int_a^b f(x,y)dxdy$

Theorem 11:

$\int\int_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy$ where $R = [a,b] \times [c,d]$

General double integrals

15.2:
To define $\int\int_D f dA$ where D bounded,let R rectangle containing D Extend f to R by defining:

Definition 1: $f^{\text{ext}}(x,y) = \begin{cases} f(x,y) \text{ if } (x,y) \in D \\ 0 \text{ if } (x,y) \notin D \end{cases}$

Defintion 2: We define $\int\int_D f dA$ to be $\int\int_R f^{\text{ext}} dA$

Elementary regions in R2

Type	1	2
Definition	$g_1 \& g_2$ continuous, but need not to be defined by single formula	$h_1 \& h_2$ continuous need not to be defined by single formula
Region D	$D = \{(x,y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$	$D = \{(x,y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$
integral	$\int\int_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$	$\int\int_D f(x,y)dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)dxdy$
	Definition 3	**Definition 4**

ANNULUS: Region between two circles.

Properties double integrals:

Property 5:
$\int\int_D [f(x,y) + g(x,y)]dA = \int\int_D f(x,y)dA + \int\int_D g(x,y)dA$

Property 6:
for constant c we have $\int\int_D cf(x,y)dA = c\int\int_D f(x,y)dA$

Property 7:
If $f(x,y) \geq g(x,y)$ for all $(x,y) \in D$:
$\int\int_D f(x,y)dA \geq \int\int_D g(x,y)dA$

Property 8:
If $D = D_1 \cup D_2$ such that D_1 and D_2 does not overlap then:
$\int\int_D f(x,y)dA = \int\int_{D_1} f(x,y)dA + \int\int_{D_2} f(x,y)dA$

Property 9:
$\int\int_D 1dA = A(D)$

Property 10:
if $m \leq f(x,y) \leq M$ for all $(x,y) \in D$:
$m \cdot A(D) \leq \int\int_D f(x,y)dA \leq M \cdot A(D)$

Lecture 10:

Rewrite a function to polar coordinates by:
$r^2 = x^2y^2$ and $x = r\cos(\theta)$ and $y = r\sin(\theta)$
Definition 2:
f continuous on polar rectangle R given by $0 \leq a \leq r \leq b$ and $\alpha \leq \theta \leq \beta$ where $0 \leq \beta - \alpha \leq 2\pi$
$\int\limits_R\int f(x,y)dA = \int\limits_\alpha^\beta \int\limits_a^b f(r\cos(\theta), r\sin(\theta))rdrd\theta$
Theorem 3:
If f continuous on polar region $D = \{(r,\theta)|\alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$ then:
$\int\limits_D\int f(x,y)dA = \int\limits_\alpha^\beta \int\limits_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))rdrd\theta$
Example:

1:
$x^2 + y^2 = 4$ so then $f(x,y) = x^2 + y$
$\int\limits_D\int f(x,y)dA = \int\limits_0^{\frac{\pi}{2}} \int\limits_0^2 (r^2\cos^2(\theta) + r\sin(\theta))rdrd\theta$
$= \int\limits_0^{\frac{\pi}{2}} \frac{1}{4}r^4\cos^2(\theta) + \frac{1}{3}r^2\sin(\theta)|_{r=1}^{r=2}d\theta = \int\limits_0^{\frac{\pi}{2}}(4\cos^2\theta + \frac{8}{3}\sin\theta)d\theta = 2(\cos\theta\sin\theta + \theta - \frac{4}{3}\cos\theta)|_0^{\frac{\pi}{2}} = \pi + \frac{8}{3}$

Applications:

Whole paragraph 15.4 is about this:
(a) Density
(b) electric charge
(c) moment (of inertia)
(d) radius of gyration of a lamina
(e) Probability
(f) Joint density function
(g) Expected values (X-mean and Y-mean)

Surface area:

Paragraph 15.5: Surface area area of a surface **Definition 1:** $A(S) = \lim\limits_{m,n\to\infty} \sum\limits_{i=1}^{m}\sum\limits_{i=1}^{n} \Delta T_{ij}$
Definition 2 and 3: if $z = f(x,y)$ where $(x,y) \in D$ and $f_x \& f_y$ continuous:
$A(s) = \int\limits_D\int \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}dA = \int\limits_D\int \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2dA}$
Paragraph 15.6:

Triple integrals:

Definition 1: simples case $B = \{(x,y,z)|a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$
Definition 2: Triple Riemann sum: $\sum\limits_{i=1}^{l}\sum\limits_{j=1}^{m}\sum\limits_{k=1}^{n}(x_{ijk}^\star, y_{ijk}^\star, z_{ijk}^\star)\Delta V$
Definition 3: Triple integral is equal to: $\int\int\limits_B\int f(x,y,z)dV = \lim\limits_{l,m,n\to\infty}\sum\limits_{i=1}^{l}\sum\limits_{j=1}^{m}\sum\limits_{k=1}^{n}(x_{ijk}^\star, y_{ijk}^\star, z_{ijk}^\star)\Delta V$

exists.
Fubini's theorem for triple integrals, theorem 4:
If f continuous on $B = [a,b] \times [c,d] \times [p,q]$ then $\int\int\int_B f dV = \int_a^b \int_c^d \int_p^q f(x,y,z) dz dy dx =$ five other orders
Definition 6: $\int\int\int_E f(x,y,z)dV = \int\int_D [\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz]dA$
Definition 7: If porjection D of E onto $xy-$ plane of type 1:
$\int\int\int_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx$
Definition 8: If projection D of E onto $xy-$ plane of type 2:
$\int\int\int_E f(x,y,z)dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdxdy$
The second part of this paragraph is about applications.

Example:

W is a graph like a icecream cone.
W =region above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $z = \sqrt{1-x^2-y^2}$
$\int\int\int_W f(x,y,z)dV = \int\int_D \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dzdA$
Boundary of shadow D by $\sqrt{x^2+y^2} = \sqrt{1-x^2-y^2} \Leftrightarrow x^2+y^2 = 1-x^2-y^2$ so D is disk of radius $\frac{1}{\sqrt{2}}$
$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)dzdydx$

Other types of coordinates:

Name	$x =$	$y =$	$z =$	$r \& \rho =$		extra
Cylindrical	$r\cos(\theta)$	$r\sin(\theta)$	z	$\tan(\theta) = \frac{y}{x}$	**15.7:Definition 1**	from polar
system			z	$\sqrt{x^2+y^2}$	**15.7:Definition 2**	From rectangular
Spherical	$\rho\sin\phi\cos\theta$	$\rho\sin\phi\sin\theta$	$\rho\cos\phi$	$\sqrt{x^2+y^2+z^2}$	**15.8:Definition 1,2**	

Corresponding integrals:

15.7:Definition 4:
$\int\int\int_E f(X,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta), r\sin(\theta))}^{u_2(r\cos(\theta), r\sin(\theta))} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta$
Where $D = \{(r\theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$ and $E = \{(x,y,z) | (x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)\}$
15.8:Definition 3 $\int\int\int_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi d\rho d\theta d\phi$
Where $E = \{(\rho,\theta,\phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$

Lecture 11:

Change of variables: double integrals:

paragraph 15.9:
Definition 1,2:
$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du = \int_c^d f(x(u))\frac{dx}{du}du$ where $x = g(u)$ and $a = g(c)$ and $b = g(d)$
Definition 7: JACOBIAN of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is:
$\frac{\partial(x,y)}{\partial(uv)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$
Definition 9: and after a lot of computations: If we have a map $T : D^\star \to D$ (so from one map to another map) and T bijective and C^1 Then $f : D \to \mathbb{R}$ integrable then substitution rule:
$\int\int_D f(x,y)dxdy = \int\int_{D^\star} f(x(u,v),y(u,v))\left|\frac{\delta(x,y)}{\delta(u,v)}\right|dudv$

Example

$T : (r\theta) \to (x(r,\theta), y(r,\theta)) = (r\cos(\theta), r\sin(\theta))$
Then $\frac{\delta(x,y)}{\delta(u,v)} = r$

So $\int\int_D f(x,y)dxdy = \int\int_{D^\star} f(r\cos(\theta), r\sin(\theta))rdrd\theta$

Change of variables: triple integrals:

When we have T one-to-one transformation maps region S in uvw space onto region R in xyz-space by: $x = g(u,v,w)$ and $y = h(u,v,w)$ and $z = k(u,v,w)$ then:
JACOBIAN: $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$ and **Definition 13:**
$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw$

Example:

$x = \rho\sin(\phi)\cos(\theta)$ and $y = \rho\sin(\phi)\sin(\theta)$ and $z = \rho\cos(\phi)$
$\frac{\delta(x,y,z)}{\delta(\rho,\phi,\theta)} = \rho^2\sin(\phi)$
So $\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W^\star} f\rho^2\sin\phi d\rho d\theta d\phi$

Vector calculus:

16.1:
Definition 1: VECTOR FIELDS: $D \subset \mathbb{R}^n$ and $F : D \mapsto \mathbb{R}^n$ then this function F is called a vector field.
Definition 2: $E \subset \mathbb{R}^3$ then vector field on $\mathbb{R}^3$ is function $\mathbf{F}$ that assigns each $(x,y,z) \in E$ in three-dimensional vector $\mathbf{F}(x,y,z)$
After this, there are a lot of examples.
GRADIENT VECTOR FIELD/CONSERVATION: $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ if there exists $f : D \to \mathbb{R}$ s.t. $F = \nabla f$

So $\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$ in $\mathbb{R}^2$
In this case f is called POTENTIAL FUNCTION for F

Line integrals:

16.2:
Definition 1: We start with C given by $x = x(t), y = y(t)$ where $a \leq t \leq b$
SMOOTH CURVE:
C smooth curve in $\mathbb{R}^n$ with parameter $\mathbf{r} : [a,b] \to \mathbb{R}^n$ and $t \mapsto \mathbf{r}(t)$
With $\mathbf{r}'(t) \neq 0$ for all $t \in [a,b]$
Then length of C given by $L = \int_a^b \|\mathbf{r}'(t)\|dt = \int_0^L ds$
Where S is called the arclength, where $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$
So $s(t) = \int_a^t \|\mathbf{r}'(\tau)\|d\tau$
Definition 2: if f smooth curve C then the line integral of f along C is $\int_C f(x,y)ds = \lim_{n\to\infty} f(x_i^\star, y_i^\star)\Delta s_i$ if the limit exist. (w.r.t arclength)
Definition 3: $\int_c f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$

C is called piecewise smooth iff C is an union of finitely many smooth curves C_i where $i = 1,\ldots,n$ s.t. the initial point of C_i equals the endpoint of C_{i-1} where $i = 2,\ldots,n$

Then $\int_C f ds := \sum_{i=1}^n \int_{C_i} f ds$
Definition 7a line integral w.r.t x $\int_c f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt$
Definition 7b line integral w.r.t y $\int_c f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$
Definition 8: When we have a line that starts at $\mathbf{r}_0$ and $\mathbf{r}_1$ then we have $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$ where $0 \leq t \leq 1$
Definition 9: LINE INTEGRALS IN SPACE: $\int_c f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt$

Lecture 12:

Line integrals

Definition 13: $\mathbf{F}$ continuous vector field, defined smooth curve C given by $\mathbf{r}(t), a \leq t \leq b$. Then LINE INTEGRAL OF F ALONG C: $\int_a^b F(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_0^L F \cdot T ds$

When $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ we have:

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Pdx + Qdy + Rdz$

Example:

F force field, then the line integral of F along the curve C is the work required to move a particle along C

$\mathbf{r} : [0,1] \to \mathbb{R}^3$—,where $\mathbf{r}(t) = t\mathbf{i} + 3t^2\mathbf{j} + 2t^2\mathbf{k}$

$F(x, y, z) = x^3\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$

$\int_C F d\mathbf{r} = \int_0^1 F(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_0^1 (t^3\mathbf{i} + (3t^2)^2\mathbf{j} + 2t^3\mathbf{k}) \cdot (\mathbf{i} + 6t\mathbf{j} + 6t^2\mathbf{k})dt = \int_0^1 (t^3 + 54t^5 + 12t^4)dt = \frac{1}{4} + 11 = 11\frac{1}{4}$

Orientation of a curve:

16.3:

Theorem 1: $\int_a^b F'(x)dx = F(b) - F(a)$ (part 2 of fundamental theorem of caluclus)

Theorem 2: C smooth curve given by $\mathbf{r}(t)$ where $a \leq t \leq b$ then:

$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$

Theorem 3: $\mathbf{F} \cdot d\mathbf{r}$ independent of path in D iff $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path in C

Theorem 4: Fundamental theorem of line integrals:

Suppose $\mathbf{F}$ continuous open connected D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ independent of path in D then $\mathbb{F}$—,conservative vector field on D that is, there exists a function f s.t. $\nabla f = \mathbf{F}$

PROOF:

Let $f(x, y) = \int_{(a,b)}^{(x,y)}$ after few computation we see that $\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x}\mathbf{F} \cdot d\mathbf{r}$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ we see that $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} Pdx + Qdy$ then $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \nabla f$

(for full proof see page 1147)

Lecture 13:

Theorem 5:
F continuous vector field. F independent of path $\Leftrightarrow \oint F \cdot d\mathbf{r} = 0$ for all closed curves C
$\oint$ stands for the integral on a closed curve.
PROOF:
$\Rightarrow$
Let C be closed curve. Then we have $\oint\limits_{C} F \cdot d\mathbf{r} = \int\limits_{c_1} F \cdot d\mathbf{r} + \int\limits_{c_2} F \cdot d\mathbf{r} = \int\limits_{-C_1} F \cdot d\mathbf{r} + \int\limits_{C_2} F \cdot d\mathbf{r} = 0$
As $-C_1$ and C_2 have the same initial and final points and F is independent of path.
$\Leftarrow$
Let C be the closed curve which is union of C_1 and C_2
$0 = \int\limits_{C} F \cdot d\mathbf{r} = \int\limits_{C_1} F \cdot d\mathbf{r} + \int\limits_{-C_2} F \cdot d\mathbf{r} = \int\limits_{C_1} F \cdot d\mathbf{r} - \int\limits_{C_2} F \cdot d\mathbf{r}$
So $\int\limits_{C_1} F \cdot d\mathbf{r} = \int\limits_{C_2} F \cdot d\mathbf{r}$ which is exactly what we wanted to show.
Definition:
A domain is called SIMPLY CONNECTED if it is connected and all closed curves in D can be contracted to a point.
Theorem 6:
Let $F = P\mathbf{i} + Q\mathbf{j}$ be a factor field on simply connected domain $D \in \mathbb{R}^2$ with $P\&Q$ being C^1
Then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftrightarrow F$ is conservative.
Paragraph 16.4:

Green's theorem:

Let D bounded domain in $\mathbb{R}^2$ with boundary Notation: ∂D consist of finitely many simple chose piece-wise C^1 curves
Orient ∂D s that D is on the left as one traverses ∂D

Let $F = P\mathbf{i} + Q\mathbf{j}$ be a C^1 Vector field on D
Then $\oint\limits_{\partial D} Pdx + Qdy = \int\int\limits_{D} (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA$
Relates line integrals to double integrals.
LHS might help to compute RHS or vica versa.
PROOF:
There is a really long proof in the book
Theorem 5: The Green's Theorem gives the following formulas for the area of D:
$A = \oint\limits_{C} xdy = -\oint\limits_{C} ydx = \frac{1}{2}\oint\limits_{C} xdy - ydx$

Curl and divergence

Paragraph 16.5:
Definition 1: Curl: curl$\mathbf{F} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{partialz})\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j}) + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}$
Remember: $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$
Definition 2: curl$\mathbf{F} = \nabla \times \mathbf{F}$
Theorem 3: if f function 3 variables, continuous second order partial derivatives then curl$(\nabla f) = \mathbf{0}$
Proof:
$\text{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = (\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y})\mathbf{i} + (\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z})\mathbf{j} + (\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x})\mathbf{k} =$
$0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$
Definition 9: div$\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ where div$\mathbf{F}$ stands for the diverengence of $\mathbf{F}$
Definition 10: div$\mathbf{F} = \nabla \cdot \mathbf{F}$
Theorem 11: if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ vector field on $\mathbb{R}^3$ and P, Q, R continuous second order partial derivatives, then div curl$\mathbf{F} = 0$
Proof:
use div curl $\mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F})$
Laplace operator: $\nabla^2 = \nabla \cdot \nabla$ name comes from relation to Laplace's equation: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$
Definition 12: Rewrite Green's theorem in vector form: $\oint\limits_C \mathbf{F} \cdot d\mathbf{r} = \oint\limits_C \mathbf{F} \cdot \mathbf{T} ds = \int\int\limits_D (\text{Curl}\,\mathbf{F}) \cdot \mathbf{k} dA$
Definition 13: or: $\oint\limits_C \mathbf{F} \cdot \mathbf{n} ds = \int\int\limits_D \text{div}\,\mathbf{F}(x, y) dA$ where $\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$

Lecture 14:

16.6:
Let **r** vector function of two parameters
definition 1: so $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$
Definition 2: PARAMETERIC EQUATIONS: $x = x(u,v), y = y(u,v)$ and $z = z(u,v)$
D is the region in the $uv-$ plane where $\mathbf{r}(u,v)$ is defined.
PARAMETERIC SURFACE: the set of all points (x,y,z) in $\mathbb{R}^3$ that satisfies the second definition and where (u,v) varies throughout D
GRID CURVE: a curve of $\mathbf{r}(u,v)$ where we have on of the parameters as a constant.

SURFACE OF REVOLUTION: surface that exists by rotating the curve $u = f(x)$ where $a \leq x \leq b$ about the $x-$ axis, where $f(x) \geq 0$
If (x,y,z) a point on this surface S then:
Definition 3: $x = x$,$y = f(x)\cos(\theta)$ and $z = f(x)\sin(\theta)$ where θ the angle of rotation.
So domain is equal to:
$a \leq x \leq b$ and $0 \leq \theta \leq 2\pi$

Tangent plane:

The partial derivatives of $\mathbf{r}(u,v)$:
Definition 4: $\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0,v_0)\mathbf{k}$
Definition 5: $\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0,v_0)\mathbf{k}$
if $\mathbf{r}_u \times \mathbf{r}_v$
$neq0$ for all values, then the surface S is SMOOTH
TANGENT PLANE: contains $\mathbf{r}_u \& \mathbf{r}_v$ and the vector $\mathbf{r}_u \& \mathbf{r}_v$ are normal vector to the tangent plane.

Definition 6: S smooth curve, given by $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ where $(u,v) \in D$
S covered just once (u,v) through domain D then SURFACE AREA:
$A(S) = \int\int_D |\mathbf{r}_u \times \mathbf{r}_v| dA$ where $\mathbf{r}_u \& \mathbf{r}_v$ like above.

Special case:

$x = x$ and $y = y$ and $z = f(x,y)$ then $\mathbf{r}_x = \mathbf{i} + (\frac{\partial f}{\partial x})\mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} + (\frac{\partial f}{\partial y})\mathbf{k}$ then
Definition 7: $\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$
So **Definition 8:** $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1} = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}$
Definition 9: so the surface area formula will become: $A(S) = \int\int_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} dA$

Surface integrals:

16.7:
Definition 1: SURFACE INTEGRAL OF f OVER THE SURFACE S by the riemann sum: $\int\int)Sf(x,y,z)dS = \lim_{m,n\to\infty} f(P_{ij}^{\star})\Delta S_{ij}$
Definition 2: $\int\int_S f(x,y,z)dS = \int\int_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v| dA$

Definition 4: $\int\int\limits_S f(x,y,z)dS = \int\int\limits_D f(x,y,g(x,y))\sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}dA$

Oriented surface:

Two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ where $\mathbf{n}_2 = -\mathbf{n}_1$
ORIENTED SURFACE: if it is possibl eot choose $\mathbf{n}$ at every (x,y,z) s.t. ,$\mathbf{n}$ varies continuously over S.
When we choose such an $\mathbf{n}$,it gives S ORIENTATION.

Definition 5: for a surface $z = g(x,y)$ we can say that: $\mathbf{n} = \frac{-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial}\mathbf{j} + \mathbf{k}}{\sqrt{1+(\frac{\partial g}{\partial x})^2+(\frac{\partial g}{\partial y})^2}}$
$\mathbf{k} > 0$ so upward orientation. If S smooth then $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$

Flux:

Let $\mathbf{n}$ normal vector ,$\rho(x,y,z)$ destiny and $\mathbf{v}(x,y,z)$ velocity field then the rate of flow per unit is given by $\rho\mathbf{v}$
If we divide S into small paches S_{ij} we obtain that the mass of fluid per unit time crossing S_{ij} in the direction of $\mathbf{n}$ is equal to: $(\rho\mathbf{v}\cdot\mathbf{n})A(S_{ij})$
So therefore we know after some steps that:
Definition 6: $\int\int\limits_S \rho\mathbf{v}\cdot\mathbf{n}dS = \int\int\limits_S \rho(x,y,z)\mathbf{v}(x,y,z)\cdot\mathbf{n}(x,y,z)dS$
If we write $\mathbf{F} = \rho\mathbf{v}$ we obtain $\int\int\limits_S \mathbf{F}\cdot\mathbf{n}dS$
Definition 8: $\mathbf{F}$ cont.vector field defined on S with unit normal vector $\mathbf{n}$ then the SURFACE INTEGRAL OF $\mathbf{F}$ OVER S IS EQUAL TO:
$\int\int\limits_S \mathbf{F}\cdot d\mathbf{S} = \int\int\limits_S \mathbf{F}\cdot\mathbf{n}dS$ This integral is also called FLUX of $\mathbf{F}$ across S
Definition 9: $\int\int\limits_S \mathbf{F}\cdot d\mathbf{S} = \int\int\limits_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)dA$
This assumes that orientation induced by $\mathbf{r}_u\times\mathbf{r}_v$. Opposite orientation?Multiply with -1

If we use $z = g(x,y)$ we see that:
Definition 9: $\mathbf{F}\cdot(\mathbf{r}_x\times\mathbf{r}_y) = (P\rangle + Q\mathbf{j} + R\mathbf{k})\cdot(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{i} + \mathbf{k})$
So then **definition 10:** $\int\int\limits_S \mathbf{F}\cdot d\mathbf{S} = \int\int\limits_D (-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R)dA$
upward orientation of S. otherwise multiply with -1

Application:

1: $\mathbf{E}$ is elictric field, then $\int\int\limits_S \mathbf{E}\cdot d\mathbf{S}$ is ELECTRIX FLUX OF E THROUGH S.
Definition 10: GAUSS'S LAW: $Q = \varepsilon_0\int\int\limits_S \mathbf{E}\cdot d\mathbf{S}$
Q is the net charge enclosed by a closed S, ε_0 is a constant(permittivity of free space)

2: $u(x,y,z)$ temperature body at (x,y,z) then heat flow: $\mathbf{F} = -K\nabla u$ K is constant called conductivity. Rate of heat flow across the surface S in the body: $\int\int\limits_S \mathbf{F}\cdot d\mathbf{S} = -K\int\int\limits_S \nabla u\cdot d\mathbf{S}$

Lecture 15:

16.8:
POSITIVE ORIIENTATION OF THE BOUNDARY CURVE C if you "walk" in positive direction around C with head pointing direction $\mathbf{n}$ then surface will be on your left.
Stokes' theorem: S oriented piecewiese-smooth surface bounded by simple,closed,piecewise-smooth C with positive orientation.
$\mathbf{F}$ vector field, components has continuous partial derivatives on open region $\mathbb{R}^3$ and $S \in \mathbb{R}^3$ then:
$\int\limits_C \mathbf{F} \cdot d\mathbf{r} = \int\int\limits_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$

Definition 1: $int \int\limits_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \int\limits_{\partial S} \mathbf{F} \cdot d\mathbb{R}$
Where ∂S—,is the positvely oriented boundary curve of the oriented surface S

Definition 3: if S_1 and S_2 oriented surface,same oriented boundary curve C, both satisfy Stoke's theorem then:
$\int\int\limits_{S_1} \text{curl}\mathbf{F} \cdot d\mathbf{S} = \int\limits_C \mathbf{F} \cdot d\mathbf{r} = \int\int\limits_{S_2} \text{curl}\mathbf{F} \cdot d\mathbf{S}$
$\mathbf{v}$: the velocity field in fluid flow.
The line integral $\int\limits_C \mathbf{v} \cdot d\mathbf{r} > 0$ then positive circulation (and otherwise negative, obviously).
We see htat $\int\limits_{C_a} \mathbf{v}\cdot d\mathbf{r} = \int\int\limits_{S_a} \text{curl}\mathbf{v}\cdot d\mathbf{S} = \int\int\limits_{S_a} \text{curl}\mathbf{v}\cdot\mathbf{n}dS \approx \int\limits_{S_a} \text{curl}\mathbf{v}(P_0)\cdot\mathbf{n}(P_0)dS = \text{curl}\mathbf{v}(P_0)\cdot\mathbf{n}(P_0)\pi a^2$
We see that $P_0(x_0, y_0, z_0)$ a point in the fluid, and S_a small disk with radius a and centered at P_0 whne $a \to 0$:
Definition 4: $\text{curl}\mathbf{v}(P_0) \cdot \mathbf{n}(P_0) = \lim\limits_{a\to 0} \frac{1}{\pi a^2} \int\limits_{C_a} \mathbf{v} \cdot d\mathbf{r}$

The divergence theorem:

16.9:
Definition 1: $\int\int\limits_S \mathbf{F} \cdot \mathbf{n}dS = \int\int\limits_E\int \text{div}\mathbf{F}(x, y, z)dV$
Divergence theorem: E simple solid region and S boundary surface E given wiht positive outward orientation.$\mathbf{F}$ vector field, with component functions continuous partial derivatives on open region containing E
Then: $\int\int\limits_S \mathbf{F} \cdot d\mathbf{S} = \int\int\limits_E\int \text{div}\mathbf{F}dV$

Assume a region E closed by the surace S_1 and S_2 where S_1 lies inside S_2
$\mathbf{n}_1$&$\mathbf{n}_2$ outward normals S_1&S_2 then boundary surface of E is $S = S_1 \cup S_2$ and $\mathbf{n} = -n_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2
Then we receive: **Definition 7:**
$\int\int\limits_E\int div\mathbf{F}dV = \int\int\limits_S \mathbf{F}\cdot d\mathbf{S} = \int\int\limits_S \mathbf{F}\cdot\mathbf{n}dS = \int\int\limits_{S_1} \mathbf{F}\cdot(-\mathbf{n}_1)dS + \int\int\limits_{S_2} \mathbf{F}\cdot\mathbf{n}_2dS = -\int\limits_{S_1}\int \mathbf{F}\cdot d\mathbf{S} + \int\limits_{S_2}\int \mathbf{F}\cdot d\mathbf{S}$

Application:

1:
We know that $\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3}\mathbf{x}$ where Q electric charge at origin,$\mathbf{x} = \langle x, y, z\rangle$ and $\mathbf{E}$ electric field.
Then we see that the electirc flux through any closed S ecloses the origin is $\int\int_S \mathbf{E} \cdot d\mathbf{S} = 4\pi\varepsilon Q$

Definition 8: $\int\int_E\int div\mathbf{E}dV = -\int\int_{S_1} \mathbf{E} \cdot d\mathbf{S} + \int\int_S \mathbf{E} \cdot dS$
(like definition 3 of 16.8)
And because we see that $div\mathbf{E} = 0$ we now that $\int\int_S \mathbf{E} \cdot d\mathbf{S} = \int\int_{S-1} \mathbf{E} \cdot d\mathbf{S}$

2:
When we have $\mathbf{F} = \rho\mathbf{v}$ so the rate of flow per unit area ,$P_0(x_{0,0}, z_0)$ a point in the fluid, and B_0 ball with center P_0 and radius a then $div\mathbf{F}(P) \approx div]\mathbf{F}(P_0)$ for all points in P in B_a since $div\mathbf{F}$ continuous.

Flux over the boundary sphere S_a:
$\int\int_{S_a} \mathbf{F} \cdot d\mathbf{S} = \int\int_{B_a}\int div\mathbf{F}dV \approx \int\int_{B_a}\int div\mathbf{F}(P_0)dV = div\mathbf{F}(P_0)V(B_a)$
When $a \to 0$ suggest **Definition 8:** $div\mathbf{F}(P_0) = \lim_{a\to 0} \frac{1}{V(B_a)} \int\int_{S_a} \mathbf{F} \cdot d\mathbb{S}$

$div\mathbf{F}(P_0)$ net rate of outward flux per unit volume at P_0 (reason name divergence).
If $div\mathbf{F}(P) > 0$: SOURCE if $div\mathbf{F}(P) < 0$: SINK